A Logic-Based Approach to Admissibility

Jeff Helzner (Columbia)

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Ordering

Myth: *The rational agent's choices are, up to picking, determined by a weak order on the space of alternatives.*

The assumption is held by well-known, normative rules such as the following:

- expected utility theories [Sav72]
- Γ-maximin [GS82, GS89]
- restricted Bayes-Hurwicz [Ell61]
- decision criteria for cases of "complete ignorance" [LR89]
  - maximin
  - pessimism-optimism

Note: The assumption is also held by descriptive theories such as prospect theory [KT79] and some versions of satisficing [Rub06].

Uncertainty

Myth: *Probability measures provide an adequate representation of credal states, at least in the case of rational agents.*

Doubts *Indeterminate* probabilities [Kyb68, Lev74].

Myth: *Decision making under uncertainty is reducible to decision making under risk through the introduction of subjective probabilities, at least in the case of rational agents.*

Doubts "Uncertainties that are not risks" [Ell61].

Examples of decision criteria that take uncertainty seriously:

- Γ-maximin
- restricted Bayes-Hurwicz
- E-admissibility [Lev74]

Decisions without ordering

In the case of Levi's decision theory, indeterminate probabilities lead to violations of the ordering assumption [Lev74, Sei88].

The ordering assumption has also been questioned in other contexts, e.g.:

- Social choice [Mou85]
- Value conflict [Lev86].
- Menu dependence [Sen02].
- Some forms of satisficing [Rub06].
- Attribute weighting [Hel09].

Choice functions

Set-valued choice functions provide a framework that is neutral with respect to ordering.

Alternatives : X is a nonempty set of alternatives.

Menus : $\mathcal{P}_\omega(X)$ is the set of all finite, nonempty subsets of X.

Admissibility : $C : \mathcal{P}_\omega(X) \to \mathcal{P}_\omega(X)$, where $C(Y) \subseteq Y$ for all $Y \in \mathcal{X}$.

Reduction to preference

Admissibility reduces to preference optimization just in case the following conditions are satisfied [Sen71]:

α : If $x \in Y \subseteq Z$ and $x \in C(Z)$, then $x \in C(Y)$.

β : If $x, y \in C(Y)$, $Y \subseteq Z$, and $y \in C(Z)$, then $x \in C(Z)$.

Some issues to consider

Dropping the ordering assumption marks a fundamental shift.

- What does *admissibility* mean when α or β fails to hold?
  - elicitation
  - testing descriptive theories
- Representation and calculation?
  - techniques from mathematics
  - techniques from logic

Syntax

Let Ω be a countable set of atoms. The *language* L (over Ω) is defined by the following inductive clauses:

- **Atoms** $\Omega \subseteq L$
- **Negation** If $\phi \in L$, then $\neg\phi \in L$
- **Conjunction** If $\phi, \psi \in L$, then $(\phi \wedge \psi) \in L$
- **Admissibility** If $\phi, \psi_1, ..., \psi_n \in L$, then $A(\phi \mid \psi_1, ..., \psi_n) \in L$
- **Necessity** If $\phi \in L$, then $\Box\phi \in L$

Semantics

A *frame* is a tuple $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W} \rangle$ that satisfies the following requirements:

F1 W is a nonempty set

F2 R is a binary relation on W.

F3 $\mathcal{X}$ is a nonempty subset of $\mathcal{P}(W)$

F4 C_w is a choice function on $\mathcal{P}_\omega(\mathcal{X}_w)$ for all $w \in W$, where

$$\mathcal{X}_w = \{Y \mid Y \neq \emptyset \text{ and } Y = S_w \cap Z \text{ for some } Z \in \mathcal{X}\}$$

and where, for all $w \in W$, $S_w = \{x \mid (w, x) \in R\}$.

Semantics

Frame conditions continued.

F5 $\mathcal{X}$ is closed under the following operations:

- $U \mapsto W - U$
- $(U, V) \mapsto U \cap V$
- $U \mapsto \{w \mid S_w \subseteq U\}$
- $(U, V_1, ..., V_n) \mapsto$

$$\{w \mid (S_w \cap U) \in C_w(\{S_w \cap V_1, ..., S_w \cap V_n\})\}$$

Interpretations

An *interpretation* of L is a frame $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W} \rangle$ along with a function π from Ω to $\mathcal{X}$. π is extended to a function π^* on L according to the following inductive clauses:

Atoms	$\pi^*(\phi) = \pi(\phi)$
Negation	$\pi^*(\neg\phi) = W - \pi^*(\phi)$
Conjunction	$\pi^*(\phi \wedge \psi) = \pi^*(\phi) \cap \pi^*(\psi)$
Necessity	$\pi^*(\Box\phi) = \{w \mid S_w \subseteq \pi^*(\phi)\}$
Admissibility	$\pi^*(A(\phi \mid \psi_1, ..., \psi_n)) =$

$$\{w \mid (S_w \cap \pi^*(\phi)) \in C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_n)\})\}$$

Given an interpretation $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi \rangle$, we write $(\mathcal{I}, w) \models \phi$ just in case $w \in \pi^*(\phi)$ and write $\mathcal{I} \models \phi$ just in case $(\mathcal{I}, w) \models \phi$ for all $w \in W$.

Basic Axioms

$$\mathbf{K}: \ (\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi$$

$$\mathbf{C1}: \ A(\psi_1 \mid \psi_2, ..., \psi_n) \rightarrow \bigwedge_{i=1}^{n} \Diamond\psi_i$$

$$\mathbf{C2}: \ \bigwedge_{i=1}^{n} \Diamond\psi_i \rightarrow \bigvee_{i=1}^{n} A(\psi_i \mid \psi_1, ..., \psi_n)$$

$$\mathbf{C3}: \ A(\phi \mid \psi_1, ..., \psi_n) \rightarrow \bigvee_{i=1}^{n} \Box(\phi \leftrightarrow \psi_i)$$

$$\mathbf{C4}: (\Box(\phi \leftrightarrow \phi') \wedge \bigwedge_{i=1}^{n} \Box(\psi_i \leftrightarrow \psi_i') \wedge A(\phi \mid \psi_1, ..., \psi_n)) \rightarrow A(\phi' \mid \psi_1', ..., \psi_n')$$

Additional Axioms

$$\mathbf{P}: \ \Diamond\top$$

$$\mathbf{C}_\alpha: \ (A(\phi \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n) \wedge \bigvee_{i=1}^{m} \Box(\phi \leftrightarrow \psi_i)) \rightarrow A(\phi \mid \psi_1, ..., \psi_m)$$

$$\mathbf{C}_\beta: \ (A(\phi_1 \mid \psi_1, ..., \psi_m) \wedge A(\phi_2 \mid \psi_1, ..., \psi_m) \wedge A(\phi_1 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n)) \rightarrow A(\phi_2 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n))$$

Basic Systems

Let $\mathcal{L}$ be the system that has all tautologies in P_L as axioms and has the following inference rule:

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}\ \text{MP}$$

Let $\mathcal{C}$ be the system that extends $\mathcal{L}$ by adding all instances of **K**, **C1**, **C2**, **C3**, and **C4** as axioms and adds the following inference rule:

$$\frac{\phi}{\Box\phi}\ \text{Gen}$$

Additional Systems

$\mathcal{C}_\alpha$: extends $\mathcal{C}$ by adding all instances $\mathbf{C}_\alpha$.

$\mathcal{C}_\beta$: extends $\mathcal{C}$ by adding all instances $\mathbf{C}_\beta$.

$\mathcal{C}_{\alpha,\beta}$: extends $\mathcal{C}$ by adding all instances $\mathbf{C}_\alpha$ and all instances of $\mathbf{C}_\beta$.

Finally, if $\mathcal{S}$ is a system, then let $\mathcal{S}^+$ be the system that extends $\mathcal{S}$ by adding $\mathbf{P}$.

Soundness and Completeness

$$\mathcal{S} \vdash \phi \text{ iff } \mathcal{I} \models \phi \text{ for all } \mathcal{I} \in I$$

$\mathcal{S}$	$\mathcal{I}$
$\mathcal{C}$	No constraints
$\mathcal{C}^+$	$S_w \neq \emptyset$
$\mathcal{C}_\alpha$	C_w satisfies α
$\mathcal{C}_\alpha^+$	C_w satisfies α and $S_w \neq \emptyset$
$\mathcal{C}_\beta$	C_w satisfies β
$\mathcal{C}_\beta^+$	C_w satisfies α and $S_w \neq \emptyset$
$\mathcal{C}_{\alpha,\beta}$	C_w satisfies α and β
$\mathcal{C}_{\alpha,\beta}^+$	C_w satisfies α and β and $S_w \neq \emptyset$

Applications and Future Work

- Applications to Sen's critique of internal consistency of choice, e.g. his "epistemic value of the menu" examples.
- Adding other modal operators to the language, e.g. operators for knowledge or belief.

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